ROOT POLYTOPES AND BOREL SUBALGEBRAS

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Abstract. Let Φ be a finite crystallographic irreducible root system and $\mathcal{P}_\Phi$ be the convex hull of the roots in Φ. We provide a uniform description of the polytope $\mathcal{P}_\Phi$. Assume that Φ is the set of roots of the complex finite simple Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$, and fix any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. We find a natural bijection between the set of the orbits of the faces of $\mathcal{P}_\Phi$ under the action of the Weyl group and a distinguished set of abelian ideals of $\mathfrak{b}$. By means of this bijection, we obtain a bijection between the set of the orbits of the faces and a distinguished set of irreducible subsystems of the affine root system associated with Φ.

1. Introduction

For any finite root system Φ in a Euclidean space $\mathcal{E}$, we denote by $\mathcal{P}_\Phi$ the convex hull of all the roots in Φ, and we call it the *root polytope* of Φ. In this paper we describe $\mathcal{P}_\Phi$ for all finite crystallographic irreducible Φ.

Some authors use the name root polytope with a different meaning. In particular, in [13] and [14], the author call root polytope the convex hull of the positive roots together with the origin, first introduced in [7]. We call this the *positive* root polytope and, if confusion may arise, $\mathcal{P}_\Phi$ the *complete* root polytope. In this paper we treat only the complete root polytopes and we shall extend our analysis to the positive polytopes in a forthcoming paper.

To our knowledge, there is no uniform abstract description of the root polytopes. In [1], some properties of the complete root polytopes are provided for the classical types through a case by case analysis, using the usual coordinate descriptions of the root systems. In this paper, we give a uniform description for all root types, and case free proofs.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and assume that Φ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathcal{E}$ is the real vector space $\mathfrak{h}^*_\mathbb{R} = Span_\mathbb{R}\Phi$, with the positive definite symmetric bilinear form induced by the Killing form, or some proportional form. We fix a system of simple roots Π in Φ, and denote by Φ^+ and $\mathfrak{b}$ the corresponding positive system and Borel subalgebra,

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space of α.

It is clear that the Weyl group W of Φ acts on $\mathcal{P}_\Phi$, thus, in order to describe $\mathcal{P}_\Phi$, we will describe the orbits of its faces, of all dimensions, under the action of W. In fact, we shall find a special representative of each orbit and compute its stabilizer in W. From this results, in particular, we shall obtain the f-polynomial of $\mathcal{P}_\Phi$.

With each subset I of Π, we associate a face F_I of $\mathcal{P}_\Phi$ and call the faces F_I, for all $I \subseteq \Pi$, the standard parabolic faces. We prove that each orbit of faces contains one and only one standard parabolic face. Set $V_I = F_I \cap \Phi$ and consider the associated vector space in $\mathfrak{g}$,

$$\mathfrak{i}_{V_I} = \sum_{\alpha \in V_I} \mathfrak{g}_\alpha.$$

We show that, if $I \neq \emptyset$, $\mathfrak{i}_{V_I}$ is an abelian ideal of $\mathfrak{b}$. By extension, as usual, we say that V_I is an abelian ideal in Φ^+. We show, moreover, that V_I is a principal ideal, i.e. that it has a unique minimal element. This means that $\mathfrak{i}_{V_I}$ is a principal abelian ideal of $\mathfrak{b}$. Hence, each face of $\mathcal{P}_\Phi$ corresponds to a principal abelian ideal of some Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The abelian ideals of the Borel subalgebras are a widely studied subject. It was a paper of Kostant's [12] that attracted much interest in them. There, among many other things, Kostant reports a result of D. Peterson, that became then well known as the "$2^{rk\mathfrak{g}}$ abelian ideals theorem". Kostant's applications and Peterson's proof of his own theorem that, besides the main result, yields an efficient technique for further investigations, were the motivation for an intense research activity on the abelian ideals in themselves (see, for example, [15], [16], and [5]). The basic idea in Peterson's proof is that of associating with each abelian ideal an element in the affine Weyl group associated with Φ. The elements of the affine Weyl group associated with the ideals $\mathfrak{i}_{V_I}$ have a simple explicit characterization.

The subsets V_I correspond in a natural way to subdiagrams of the extended Dynkin diagram of Φ and hence to standard parabolic subsystems of the affine root system $\widehat{\Phi}$ associated with Φ. Let $\widehat{\Pi}$ be an extension of Π to a simple system of $\widehat{\Phi}$ and $\alpha_0 \in \widehat{\Phi}$ be such that $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$. The map from subsets of Π to orbits of faces is not injective, except in rank 1 case. Thus it may happen that $F_I = F_J$ for different subsets I and J of Π. We prove that, for each standard parabolic face F, the set of $I \subseteq \Pi$ such that $F = F_I$ has a minimum and a maximum, and we give an explicit description of them. Then we prove that, if $\overline{I}_F$ is the maximum, then the root subsystem of $\widehat{\Phi}$ generated by $\widehat{\Pi} \setminus \overline{I}_F$ is irreducible. Conversely, if $\overline{I}$ is a subset of Π such that the root subsystem generated by $\widehat{\Pi} \setminus \overline{I}_F$ is irreducible, then $\overline{I}$ is the maximum of the subsets I such that $F_I = F_{\overline{I}}$. Hence, we obtain that the orbits of the faces of $\mathcal{P}_\Phi$ under the action of the Weyl group are in bijection with the irreducible standard parabolic subsystems of the affine root

system that contain the affine root. If $\overline{I} \subseteq \Phi$ is such that the root subsystem generated by $\widehat{\Pi} \setminus \overline{I}$ is irreducible, then the ideal $V_{\overline{I}}$ has a simple description as a subset of this root subsystem, which give also a simple formula for $|V_{\overline{I}}|$. Moreover, the dimension of the standard parabolic face $F_{\overline{I}}$ is $|\Pi| - |\overline{I}|$. Finally, we can characterize the stabilizer in W of the standard parabolic face $F_{\overline{I}}$ as a standard parabolic subgroup of W, corresponding to a certain subset of $\overline{I}$.

We sum up our main results in the following theorem. For each $\Gamma \subseteq \Pi$, let $\widehat{\Gamma} = \Gamma \cup \{\alpha_0\}$, $\widehat{\Phi}(\widehat{\Gamma})$ be the standard parabolic subsystem of $\widehat{\Phi}$ generated by $\widehat{\Gamma}$, and

$$\mathcal{I} = \{\Gamma \subseteq \Pi \mid \widehat{\Phi}(\widehat{\Gamma}) \text{ is irreducible}\}.$$

For all $\Gamma \in \mathcal{I}$, let

$$Adj(\Gamma) = \{\alpha \in \Pi \setminus \Gamma \mid \exists \beta \in \widehat{\Gamma} \text{ s. t. } (\alpha, \beta) \neq 0\}, \quad \text{and} \quad \Gamma^* = \Pi \setminus Adj(\Gamma).$$

Moreover, for all $J \subseteq \Pi$, let W_J be the subgroup of the Weyl group generated by the reflections with respect to the roots in J.

Theorem. *Let $\mathcal{F}$ be the set of the orbits of the action of W on the set of the faces of $\mathcal{P}_\Phi$. Then $\mathcal{F}$ is in bijection with $\mathcal{I}$.*

Assume that $\Gamma \in \mathcal{I}$, F is a face in the orbit corresponding to Γ, and $V_F = F \cap \Phi$. Then $\dim F = |\Gamma|$. If $\Gamma \neq \Pi$, $|V_F| = \frac{1}{2}|\widehat{\Phi}(\widehat{\Gamma}) \setminus \Phi(\Gamma)|$ and the vertices of F are in bijection with the positive long roots in $\widehat{\Phi}(\widehat{\Gamma}) \setminus \Phi(\Gamma)$. Moreover, the stabilizer of F in W is conjugated to W_{Γ^*}.

In particular, the f-polynomial of $\mathcal{P}_\Phi$ is

$$\sum_{\Gamma \in \mathcal{I}} [W : W_{\Gamma^*}] t^{|\Gamma|}.$$

2. Preliminaries

In this section, we fix the notation and recall the basic results that we most frequently use in the paper. For basic facts about root systems, Weyl groups, and convex polytopes, we refer the reader, to [3], [9], [2], [10], and [8].

Given $n, m \in \mathbb{Z}$, with $n \leq m$, we let $[n, m] = \{n, n+1, \dots, m\}$ and, for $n \in \mathbb{N} \setminus \{0\}$, we let $[n] = [1, n]$. For every set I, we denote its cardinality by $|I|$. We write $:=$ when the term at its left is defined by the expression at its right.

For the reader's convenience, we list our notations on root systems here.

Φ	crystallographic irreducible root system,
n	rank of Φ,
Φ_ℓ	set of long roots ($\Phi_\ell = \Phi$ in simply laced cases),
R_ℓ	$= R \cap \Phi_\ell$, for all $R \subseteq \Phi$,
$(-,-)$	scalar product on the real space of Φ,
$\vert\vert - \vert\vert$	norm from $(-,-)$,
α-(β)	α-string through β,
$\Pi = \{\alpha_1, \dots, \alpha_n\}$	set of simple roots,
$\Omega = \{\breve{\omega}_1, \dots, \breve{\omega}_n\}$	set of fundamental coweights (dual basis of Π),
Φ^+	set of positive roots,
$\Phi(\Gamma)$	standard parabolic subsystem generated by Γ
$\Phi^+(\Gamma)$	$= \Phi(\Gamma) \cap \Phi^+$
$c_i(\alpha)$	i-th coordinate of α w.r.t. Π,
$\mathrm{supp}(\alpha)$	$= \{\alpha_i \in \Pi \mid c_i(\alpha) \neq 0\}$,
$ht(\alpha)$	$= \sum_{i=1}^n c_i(\alpha)$ (height of the root α),
θ	highest root,
m_i	$= c_i(\theta)$
W	Weyl group of Φ,
ℓ	length function of W w.r.t. Π,
s_α	reflection through the hyperplane $\alpha^\perp$,
$D_r(w)$	$= \{i \in [n] \mid \ell(ws_{\alpha_i}) < \ell(w)\}$ right descent set,
w_0	longest element of W w.r.t. Π.

To simplify notation, for $\alpha \in \Phi$, we write also $\alpha > 0$ (respectively, $\alpha < 0$) instead of $\alpha \in \Phi^+$ (respectively, $\alpha \in -\Phi^+$). When an element w of the Weyl group acts on the vector α, we write $w(\alpha)$ as well as $w\alpha$.

The *root poset of* Φ (w.r.t. Π) is the partial order set whose underlying set is Φ^+ and whose order structure is defined by letting $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of roots in Φ^+. The root poset could be equivalently defined as the transitive closure of the relation $\alpha \lhd \beta$ if and only if $\beta - \alpha$ is a simple root. The root poset hence is ranked by the height function and has the highest root θ as maximum. A dual order ideal is, as usual, a subset I of Φ^+ such that, if $\alpha \in I$ and $\beta \geq \alpha$, then $\beta \in I$.

For the reader convenience, we collect in the following propositions the standard results on root systems that are frequently used in the paper.

Proposition 2.1. *Let Φ be any root system. If $\alpha, \beta \in \Phi$, $\alpha \neq -\beta$, $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$. Moreover, if L is a subset of Π which is connected in the Dynkin diagram of Φ, then $\sum_{\alpha \in L} \alpha$ is a positive root.*

Proposition 2.2 ([3], Ch. VI, §1)**.** *Let Φ be any root system, and let α and β be non-proportional roots of Φ. Then the set $\{j \in \mathbb{Z} \mid \beta + j\alpha \in \Phi\}$ is an interval $[-q, p]$ containing 0. The set α-$(\beta) = (\beta + \mathbb{Z}\alpha) \cap \Phi$ is the* α-string through β *and has exactly $-\frac{2(\gamma,\alpha)}{(\alpha,\alpha)} + 1$ roots, where $\gamma = \beta - q\alpha$ is the origin of the string.*

Proposition 2.3 ([3], Ch. VI, §1, Proposition 2.4). *Let Φ be any root system and let Φ' be the intersection of Φ with a subspace of E. Then*

(1) *Φ' is a root system in the subspace it spans;*
(2) *given any basis Π' of Φ', there exists a basis of Φ containing Π'.*

Proposition 2.4. *Let Φ be an irreducible root system. If $m_i = 1$ then $-\theta \cup (\Pi \setminus \{\alpha_i\})$ is a basis of Φ.*

Let $\mathfrak{g}$ be a simple Lie algebra of rank n and fix a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\Phi \subseteq \mathfrak{h}^*$ is the irreducible root system of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space of α. For every choice of a basis Π of Φ, we have the corresponding standard Borel subalgebra $\mathfrak{b}(\Pi) = \mathfrak{h} \oplus \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. We let $\mathfrak{b} := \mathfrak{b}(\Pi)$ if no confusion arises. Being $\mathfrak{h}$-stable, any ideal $\mathfrak{i}$ of $\mathfrak{b}$ is compatible with the root space decomposition. Moreover, since, given $\alpha, \alpha' \in \Phi^+$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}]$ is equal to $\mathfrak{g}_{\alpha+\alpha'}$ if $\alpha + \alpha' \in \Phi$ and is trivial otherwise, if $\mathfrak{i} = \sum_{\alpha\in\Gamma} \mathfrak{g}_\alpha$ is an ideal of $\mathfrak{b}$, then $\Gamma \subseteq \Phi^+$ satisfies $(\Gamma + \Phi^+) \cap \Phi \subseteq \Gamma$, or, equivalently, Γ is a dual order ideal in the root poset. If we further require that $\mathfrak{i}$ be abelian, than Γ must satisfy also the abelian condition: $(\Gamma + \Gamma) \cap \Phi = \emptyset$. Indeed, all abelian ideals of $\mathfrak{b}$ are of this kind since they must be *ad*-nilpotent (i.e., included in $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$). By a *principal abelian ideal of* $\mathfrak{b}$, we mean an abelian ideal $\mathfrak{i}$ of the form $\mathfrak{i} = \sum_{\alpha\in\Gamma} \mathfrak{g}_\alpha$, where Γ, as a subposet of the root poset, has a minimum η (hence Γ is an interval since the highest root θ is the maximum). A principal abelian ideal is generated by any non-zero vector of the root space $\mathfrak{g}_\eta$.

3. Coordinate Faces

We fix a basis Π of Φ and, with every simple root, we associate a (non necessarily maximal) face of the root polytope $\mathcal{P}_\Phi$, that we call coordinate face. The coordinate faces are shown to correspond to principal abelian ideals of the Borel subalgebra $\mathfrak{b}$ corresponding to Φ. Some results can be extended to not irreducible not reduced root system.

Recall that the highest root θ has m_i as i-th coordinate w.r.t. Π, i.e. $(\theta, \breve{\omega}_i) = m_i$. For all $i = 1, \dots, n$, we define

$$V_i := \{\alpha \in \Phi^+ \mid (\theta, \breve{\omega}_i) = m_i\}$$

and we call the convex hull of V_i the *i-th coordinate face*. We denote it by F_i. Note that the coordinate faces are actually faces of the root polytope $\mathcal{P}_\Phi$, since $\mathcal{P}_\Phi$ is included in the half-space $(-, \breve{\omega}_i) \le m_i$, and that the maximal root θ belongs to all coordinate faces. Not all coordinate faces are facets (i.e. faces of codimension 1), and indeed they can be of all dimensions and one properly included in others. We later show that the maximal coordinate faces are facets.

The following result gives a restriction on the sets of the roots lying on the same coordinate face.

Proposition 3.1. *Let $\mathfrak{g}$ be a Lie algebra having Φ as root system and let $\mathfrak{b}$ be its standard Borel subalgebra associated with Π. Then the subspaces $\mathfrak{i}_{V_i} := \sum_{\alpha \in V_i} \mathfrak{g}_\alpha$, $i = 1, \dots, n$, are distinct principal abelian ideals of $\mathfrak{b}$.*

Proof. In this proof we use Proposition 2.1 several times without explicit mention.

To show that the set $\mathfrak{i}_{V_i}$ is an ideal of $\mathfrak{b}$ we need to show that, given $\alpha \in V_i$ and $\beta \in \Phi^+$, either $\alpha + \beta \notin \Phi$, or $\alpha + \beta \in V_i$. This is clear since the linear functional $(-, \breve{\omega}_i)$ cannot take values $> m_i$ on the roots. The abelianity of the ideal $\mathfrak{i}_{V_i}$ follows by the fact that, for $\alpha, \alpha' \in V_i$, $\alpha + \alpha' \notin \Phi$ since $(\alpha + \alpha', \breve{\omega}_i) = 2m_i$.

Recall that $\mathfrak{i}_{V_i}$ is principal if the subset V_i of the root poset has a minimum. This is clear if $m_i = 1$ since $\alpha_i \in V_i$, in this case. Hence suppose $m_i > 1$ and, by contradiction, let α and α' be two distinct minimal roots in V_i. The scalar products (α, α_j) are ≤ 0, for all $j \neq i$, otherwise $\alpha - \alpha_j \in \Phi^+$ and α would not be minimal. Similarly, $(\alpha', \alpha_j) \leq 0$, for all $j \neq i$. Then $(\alpha, \alpha_i) > 0$ and $(\alpha', \alpha_i) > 0$ since there are no positive roots in the anti-fundamental Weyl chamber. Moreover, $(\alpha, \alpha') = 0$: (α, α') cannot be > 0 since α and α' are incomparable and one of $\alpha - \alpha'$ and $\alpha' - \alpha$ would be a positive root ; (α, α') cannot be < 0 since $\alpha + \alpha'$ cannot be a root being $(\alpha + \alpha', \breve{\omega}_i) = 2m_i$. Now, $\alpha - \alpha_i \in \Phi^+$ since $(\alpha, \alpha_i) > 0$ and hence also $\alpha + \alpha' - \alpha_i$ is a positive root since $(\alpha - \alpha_i, \alpha') = -(\alpha_i, \alpha') < 0$. This is a contradiction since $(\alpha + \alpha' - \alpha_i, \breve{\omega}_i) = 2m_i - 1 > m_i$.

Since the minimal root of V_i has positive scalar product only with α_i among the simple roots, all these minimal roots are distinct and hence $V_i \neq V_j$ whenever $i \neq j$. $\square$

We explicitly notice that Theorem 3.1 clearly implies that the coordinate faces F_h and F_k, $h \neq k$, cannot coincide since V_i is exactly the set of roots in F_i, for all $i \in [n]$.

Another property of the coordinate face F_i is that the barycenter of V_i is parallel to the i-th fundamental coweight $\breve{\omega}_i$. This follows by the following general lemma, which states that every α-string is centered on a vector orthogonal to α.

Lemma 3.2. *Let $\alpha, \beta, \in \Phi$ and let α-$(\beta) = (\beta + \mathbb{Z}\alpha) \cap \Phi$ be the α-string through β. Set $\mu := \sum_{\gamma \in \alpha\text{-}(\beta)} \gamma$. Then*

$$(\mu, \alpha) = 0.$$

Proof. We may suppose that β is the origin of its α-string. Then, by Proposition 2.2, α-$(\beta) = \{\beta + j\alpha \mid j = 0, 1, \dots, -\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\}$. The middle vector $\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is orthogonal to α. $\square$

Proposition 3.3. *The barycenter of the roots in the i-th coordinate face F_i is parallel to the i-th fundamental coweight:*

$$\sum_{\alpha\in V_i}\alpha = \frac{m_i|V_i|}{||\breve{\omega}_i||^2}\breve{\omega}_i.$$

Proof. By the definition of V_i, if $\alpha\in V_i$ and $\alpha\pm\alpha_j\in\Phi$ for a certain $j\neq i$, then $\alpha\pm\alpha_j\in V_i$. Hence V_i is a union of α_j-string, for all $j\neq i$. By Lemma 3.2, $(\sum_{\alpha\in V_i}\alpha,\alpha_j)=0$, for all $j\neq i$. Hence $\sum_{\alpha\in V_i}\alpha$ is a multiple of $\breve{\omega}_i$. Since $(\alpha,\breve{\omega}_i)=m_i$ for all $\alpha\in V_i$, we get the assertion. □

The following result gives a direct way to tell the dimension of the coordinate face F_i from its minimal root η_i, and is needed to show that the maximal coordinate faces are facets.

Proposition 3.4. *Let $\eta_i := \sum_{k=1}^n n_k\alpha_k$ be the minimal root of V_i. The face F_i has codimension $|\{k \mid n_k=m_k\}|$.*

Proof. In the root poset, we may find a chain $\eta_i=\gamma_0\lhd\gamma_1\lhd\cdots\lhd\gamma_t=\theta$ (here $t=\sum_{k=1}^n m_k-n_k$). All these roots are in V_i and we have $\{\gamma_i-\gamma_{i-1}\mid i\in[t]\}=\{\alpha_k\mid n_k\neq m_k\}$. This set spans a subspace E' of dimension $|\{k\mid n_k\neq m_k\}$. Any other vector obtained as a difference of two roots in V_i is in E'. □

Theorem 3.5. *The following are equivalent.*

1. *The coordinate face F_i is a facet of $\mathcal{P}_\Phi$.*
2. *The minimal root η_i of V_i satisfies $(\eta_i,\breve{\omega}_j)\neq m_j$, for all $j\neq i$.*
3. *For all $j\neq i$, there exists $\alpha\in V_i$ such that $(\alpha,\breve{\omega}_j)\neq m_j$.*
4. *The set V_i contains an α_j-string which is non-trivial (i.e. of cardinality >1), for all $j\neq i$.*
5. *F_i is maximal among the coordinate faces $\{F_j\mid j=1,\dots,n\}$.*

Proof. We prove $1\implies 2\implies 3\implies 4\implies 1$ and then $1\implies 5\implies 4$.

The assertion $1\implies 2$ follows by Proposition 3.4 and $2\implies 3$ is trivial. Fix $j\neq i$ and suppose there exists $\alpha\in V_i$ such that $(\alpha,\breve{\omega}_j)\neq m_j$. Take a chain $\alpha=\gamma_0\lhd\gamma_1\lhd\cdots\lhd\gamma_t=\theta$ in the root poset. Since $(\alpha,\breve{\omega}_j)\neq m_j$ and the root poset is ranked, there exists r such that $\gamma_{r+1}=\gamma_r+\alpha_j$. Hence the α_j-string through γ_r is non-trivial and we have $3\implies 4$. Clearly $4\implies 1$ since the difference of two consecutive roots in an α_j-string is α_j.

It is trivial that $1\implies 5$. Let us prove $5\implies 4$ by contradiction. So assume that there exists $j\neq i$ such that all α_j-strings in V_i are trivial. For every $\alpha\in V_i$, the difference of two consecutive roots in any chain in the root poset from α to θ cannot be α_j. This means that $(\alpha,\breve{\omega}_j)=m_j$ for all $\alpha\in V_i$. Hence $V_j\supseteq V_i$. Since all the coordinate faces F_k are distinct by Theorem 3.1, F_i is not maximal. □

Note that, if $m_i = 1$, then α_i coincides with the minimal root η_i of V_i, and hence F_i is a facet. So, for example, in type A, all coordinate faces F_i are facets. On the other hand, in B_n $(n \geq 3)$, C_n $(n \geq 2)$, and D_n $(n \geq 4)$ there are, respectively, only 2, 1, and 3 coordinate facets: F_1 and F_n in B_n, F_n in C_n, and F_1, F_{n-1} and F_n in D_n.

4. Parabolic faces

We say that a face F of $\mathcal{P}_\Phi$ is a *standard parabolic face* if it is an intersection of coordinate faces, i.e. if there exists $I \subseteq [n]$ such that

$$F = \cap_{i \in I} F_i.$$

We let $F_I := \cap_{i \in I} F_i$ and $V_I := \cap_{i \in I} V_i$ be the set of roots in F_I. We set $F_\emptyset := \mathcal{P}_\Phi$ and $V_\emptyset := \Phi$. Notice that the standard parabolic face F_I is in the affine subspace $\{x \mid (x, \breve{\omega}_i) = m_i, \ \forall i \in I\}$ and that V_I is a union of α_j-string, for all $j \notin I$.

Clearly, the Weyl group W acts on the set of the faces of $\mathcal{P}_\Phi$. We say that a face F is a *parabolic face* if it is transformed into a standard parabolic face by an element in W.

We generalize to the standard parabolic faces some of the result of the previous section. First, we show that the abelian ideals associated with standard parabolic faces are principal, and we compute the dimensions of the standard parabolic faces.

Theorem 4.1. *Let $\emptyset \neq I \subseteq [n]$. Then*

1. *V_I has a minimum in the root poset,*
2. *if the minimum of V_I is $\eta_I := \sum_{k=1}^n n_k \alpha_k$, then the parabolic face F_I has codimension $|\{k \mid n_k = m_k\}|$.*

Proof. We prove 1 by contradiction. Let η and η' be two distinct minimal roots in V_I. The roots η and η' are perpendicular by Proposition 2.1: both $\eta - \eta'$ and $\eta + \eta'$ cannot be roots since η and η' are incomparable and belong to a same face. Moreover, for all $k \notin I$, $(\eta, \alpha_k) \leq 0$ and $(\eta', \alpha_k) \leq 0$, because both $\eta - \alpha_k$ and $\eta' - \alpha_k$ cannot be roots since η and η' are minimal. Being positive roots, η and η' cannot be in the anti-fundamental Weyl chamber, hence at least one of the products (η, α_i), $i \in I$, must be positive, and analogously for the products (η', α_i), $i \in I$. Let α_t and $\alpha_{t'}$ be of minimal distance in the Dynkin diagram of Φ such that $(\eta, \alpha_t) > 0$ and $(\eta', \alpha_{t'}) > 0$ (hence $t, t' \in I$). Let L be the set of simple roots in the path connecting α_t and $\alpha_{t'}$ in the Dynkin diagram.

Recall that the sum of the simple roots in a connected subdiagram of the Dynkin diagram is always a root. Moreover, the support of a root is connected in the Dynkin diagram (see, for example, Corollary 3 of Proposition 19, Chapter

VI, Section 1, n. 6 of [3]). Hence $\alpha_L := \sum_{\alpha\in L}\alpha$ is a positive root, $\alpha_{L-t} := \sum_{\alpha\in L\setminus\{\alpha_t\}}\alpha$ and $\alpha_{L-t'} := \sum_{\alpha\in L\setminus\{\alpha_{t'}\}}\alpha$ are null or positive roots, and $\alpha_L < \eta, \eta'$ (η and η' are both minimal so the equalities cannot hold).

By the minimality of t and t', the only root in L with positive scalar product with η is α_t and the only root in L with positive scalar product with η' is $\alpha_{t'}$. Moreover, $(\eta, \alpha_{L-t}) \not< 0$ otherwise $\eta + \alpha_{L-t}$ would be a root such that $(\eta + \alpha_{L-t}, \breve{\omega}_{t'}) > m_{t'}$, and hence $(\eta, \alpha_{L-t}) = 0$. This implies $(\eta, \alpha_L) > 0$. Analogously, $(\eta', \alpha_L) > 0$. By the last two inequalities and the fact that $(\eta, \eta') = 0$, $\eta - \alpha_L$, $\eta' - \alpha_L$ and $\eta - \alpha_L + \eta'$ are positive roots. We have that $(\eta - \alpha_L + \eta', \breve{\omega}_r)$ is equal to $2m_r - 1$ for all $r \in R$, and to $2m_r$ for all $r \in I \setminus R$, where $R := I \cap \{r \mid \alpha_r \in L\}$. Necessarily $I = R$ and $m_r = 1$ for all $r \in I$. Hence α_L is in V_I and this is a contradiction since $\alpha_L < \eta, \eta'$.

To prove 2, we may proceed as in the proof of Proposition 3.4. □

Lemma 4.2. *Let $I \subseteq [n]$. The barycenter of the standard parabolic face F_I is in the cone generated by the coweights $\breve{\omega}_i$, $i \in I$.*

Proof. Since V_I is union of α_j-string, for all $j \notin I$, the barycenter $\sum_{\alpha\in V_I}\alpha$ of F_I is orthogonal to α_j, for all $j \notin I$, by Lemma 3.2. Hence it is in the span of the coweights $\breve{\omega}_i$, $i \in I$. Morover, by Proposition 2.1, $(\alpha, \alpha_i) \geq 0$ for all $\alpha \in V_I$ and $i \in I$, since $\alpha + \alpha_i$ cannot be a root. Hence the barycenter has nonnegative scalar product with all α_i, $i \in I$, and we have the assertion. □

Corollary 4.3. *Two distinct standard parabolic faces cannot be transformed into one another by elements in the Weyl group W.*

Proof. The barycenter of every standard parabolic face is in the closure of the fundamental Weyl chamber, by Lemma 4.2. Since the closure of the fundamental chamber is a fundamental domain for the action of W, the barycenters of two standard parabolic faces in the same W-orbit must coincide. Since distinct faces have distinct barycenters, we get the assertion. □

Given an arbitrary face F of $\mathcal{P}_\Phi$, we let $V_F := \Phi \cap F$ be the set of roots in F (so, for all $I \subseteq [n]$, $V_{F_I} = V_I$), and $E_F := \mathit{Span}\{\alpha - \alpha' \mid \alpha, \alpha' \in F\}$ be the vector subspace parallel to the smallest affine subspace containing F.

The following result is central for our investigations.

Theorem 4.4. *Let F be a face of $\mathcal{P}_\Phi$. The set V_F of the roots in F is not the union of two non-trivial orthogonal subsets.*

Proof. Fix a face F of P_Φ and, by contradiction, suppose $V_F = V' \cup V''$, where V' and V'' are non-empty sets such that every root in V' is orthogonal to every root in V''. Let $\Phi_F := \Phi \cap \mathit{Span} V_F$, $\Phi'_F := \Phi \cap \mathit{Span} V'$, and $\Phi''_F := \Phi \cap \mathit{Span} V''$.

Moreover, let $\mathcal{F}_F$ be the set of linear functional defining E_F and suppose that the smallest subspace containing F is the intersection of the hyperplanes $f(x) = m_f$, for $f \in \mathcal{F}_F$, while every functional f takes values $\leq m_f$ on Φ.

Now we want to prove that $\Phi_F = (\Phi'_F \cup \Phi''_F)$ and so that Φ_F is reducible. Suppose $\alpha \in \Phi_F \setminus (\Phi'_F \cup \Phi''_F)$. Being $\mathit{Span}V_F = \mathit{Span}V' \cup \mathit{Span}V''$, $\mathit{Span}V'$ is the orthogonal complement of $\mathit{Span}V''$ in $\mathit{Span}V_F$, and vice-versa. Hence, there exists $\beta_i \in V_i$ such that $(\alpha, \beta_i) \neq 0$, for $i = 1, 2$. We claim that $(\alpha, \beta_i) > 0$, for $i = 1, 2$. If this were not true, $\alpha + \beta_i$ would be a root (Proposition 2.1), $f(\alpha + \beta_i) = f(\alpha) + m_f$, forcing $f(\alpha) = 0$, for all $f \in \mathcal{F}_F$, and $\alpha + \beta_i \in F$, for $i = 1, 2$. Since, say, $\alpha + \beta_1$ is not perpendicular to β_2, we would have that $\alpha + \beta_1 \in V''$ is perpendicular to β_1 and we would get that $s_{\beta_1}(\alpha) = s_{\beta_1}(\alpha + \beta_1 - \beta_1) = \alpha + \beta_1 - s_{\beta_1}(\beta_1) = \alpha + 2\beta_1$, but $\alpha + 2\beta_1$ cannot be a root being $f(\alpha + 2\beta_1) = 2m_f$. The claim is proved. Choose $f \in \mathcal{F}_F$ such that $f(\alpha) < m_f$ (this exists since $\alpha \notin F$). Then $s_{\beta_1} s_{\beta_2}(-\alpha) = -\alpha + c_1\beta_1 + c_2\beta_2$, for positive integers c_1 and c_2 and this gives $f(s_{\beta_1} s_{\beta_2}(-\alpha)) > m_f$, which is impossible. Thus $\Phi_F = \Phi'_F \cup \Phi''_F$.

By Proposition 2.3, we can find 3 subsets $\overline{\Pi}, \overline{\Pi}_1, \overline{\Pi}_2 \subseteq \Phi$, which are respectively basis of Φ, Φ'_F and Φ''_F, such that $\overline{\Pi}_1, \overline{\Pi}_2 \subseteq \overline{\Pi}$. We let $\overline{\Pi}' := \overline{\Pi} \setminus (\overline{\Pi}_1 \cup \overline{\Pi}_2)$ and $\alpha := \sum_{\beta \in \overline{\Pi}'} \beta$. Since Φ is irreducible and every root in $\overline{\Pi}_1$ is orthogonal to every root in $\overline{\Pi}_2$, there exist $\beta_i \in \overline{\Pi}_i$ such that $(\alpha, \beta_i) < 0$, for $i = 1, 2$.

In the root system Φ_1 generated by the connected component of β_1 in $\overline{\Pi}_1$, there are certainly roots in V', and, analogously, in the root system Φ_2 generated by the connected component of β_2 in $\overline{\Pi}_2$, there are certainly roots in V''. For a general irreducible root system, both the long roots and the short roots (if any) generate lattices of maximum rank: hence, in Φ_i, there are roots of any possible length whose support contains β_i, for $i = 1, 2$. Since the Weyl group of an irreducible root system is transitive on the roots of same length, there exist $\gamma_i \in F_i$ and $w_i \in W(\Phi_i)$ such $w_i(\gamma_i)$ has support containing β_i, for $i = 1, 2$. As elements of the Weyl group W of the entire root system Φ, w_1 fixes Φ_2 and w_2 fixes Φ_1, by the orthogonality condition. Let $w := w_1 w_2$ and $\tilde{\alpha} := w^{-1}(\alpha)$. Since the transformations in W are isometries and $(\alpha, w(\gamma_i)) < 0$, we have $(\tilde{\alpha}, \gamma_i) < 0$, for $i = 1, 2$. Then the root $s_{\gamma_1} s_{\gamma_2}(\tilde{\alpha})$ is of the form $\tilde{\alpha} + c_1\gamma_1 + c_2\gamma_2$, for certain positive integers c_1 and c_2. Since the absolute value of every $f \in \mathcal{F}_F$ on the roots is $\leq m_f$, we have that $c_1 = c_2 = 1$ and $f(\tilde{\alpha}) = -m_f$. Hence $-\tilde{\alpha} \in F$, and this is a contradiction. $\square$

As first corollaries of Theorem 4.4, we have that subspaces spanned by faces do not disconnect the root system Φ and that the faces are parallel to subspaces spanned by roots.

Corollary 4.5. *Let F be any face of $\mathcal{P}_\Phi$. Then*

(1) *the root system $\Phi \cap SpanF$ is irreducible,*
(2) *the subspace E_F is spanned by roots in Φ*

Proof. To prove 1, by contradiction assume that $\Phi \cap SpanF = \Phi'_F \cup \Phi''_F$ and proceed as in the second part of the proof of Theorem 4.4.

Let us prove 2. Let Γ_F be the graph having F as vertex set and where $\{\beta, \beta'\} \subseteq F$ is an edge if and only if $(\beta, \beta') \neq 0$. By Proposition 2.1, if $(\beta, \beta') \neq 0$, $\beta \neq \beta'$, then $(\beta, \beta') > 0$ since the sum of two roots in a face cannot be a root, and hence $\beta - \beta'$ is a root. By Theorem 4.4, Γ_F is connected; this implies that E_F is spanned by roots: $Span(\Phi \cap E_F) = E_F$. □

We end this section with a direct description of the 1-skeleton of $\mathcal{P}_\Phi$, whose edges are, in fact, made up of roots.

Corollary 4.6. *Let F be a face of $\mathcal{P}_\Phi$ of dimension 1. Then the roots in F form a string with 2 or 3 roots: if γ and γ' are the vertices of F, then either*

(1) *γ and γ' are not perpendicular, there are no other roots in F, and $\gamma - \gamma'$ is a root, or*
(2) *γ and γ' are perpendicular, there is only a third root γ'' in F (γ'' short), and $\gamma - \gamma'' = \gamma'' - \gamma'$ is a root (so that the edge $\gamma - \gamma'$ is twice a root).*

Proof. Let $\gamma_0, \gamma_1, \dots, \gamma_r$ be the roots in F. Since F has dimension 1, we may assume that $\gamma_i = \gamma_0 + k_i\beta$, $k_i \in \mathbb{R}$, for all $i \in [r]$, with $1 = k_1 < k_2 < \cdots < k_r$, $Span\,\beta = E_F$, and $(\gamma_0, \beta) \leq 0$. Recall that two roots in a face cannot have negative scalar product. Using Theorem 4.4, we easily show that $(\gamma_i, \gamma_{i+1}) > 0$, for all $i \in [0, r-1]$. Then $\beta \in \Phi$ and $\gamma_i - \gamma_{i-1}$ is both a root and a positive multiple of β, for all $i \in [r]$, by Proposition 2.1. Then $\{\gamma_0, \gamma_1, \dots, \gamma_r\}$ is the β-string through γ_0. It is well known that the strings can have only 1,2,3, or 4 roots. In this case, it clearly does not have 1 root and it cannot have 4 roots since in a string of 4 roots the product of the first with the forth one is negative. Hence we have done, cause the string with 3 roots are necessarily of the type in the statement (see [3], Ch. VI, §1, n. 4). □

5. Uniform description of the faces

The aim of this section is to prove that all the faces of $\mathcal{P}_\Phi$ are parabolic, and, in fact, there is a bijection between the W-orbits of faces and the standard parabolic faces. As a byproduct of this result, we obtain that the standard parabolic faces of $\mathcal{P}_\Phi$ are in bijection with the connected subdiagrams of the extended Dynkin diagram of Φ that contain the affine node.

For any subset Γ of Φ, we denote by $W(\Gamma)$ the subgroup of W generated by the reflections with respect to the roots in Γ, and by $\Phi(\Gamma)$ the root system generated by Γ, i.e. $\Phi(\Gamma) := \{w(\gamma) \mid w \in W(\Gamma),\ \gamma \in \Gamma\}$. Moreover, we denote by Γ_ℓ the set of the long roots in Γ, i.e.

$$\Gamma_\ell := \Gamma \cap \Phi_\ell.$$

It is clear that $\Phi(\Gamma_\ell) \subseteq \Phi(\Gamma)_\ell$.

For any root subsystem R of Φ, we call the root system $\mathit{Span}R \cap \Phi$ the *parabolic closure* of R in Φ.

Lemma 5.1. *Let R be an irreducible root subsystem of Φ. Then the parabolic closures of R and R_ℓ coincide.*

Proof. It is clear that the parabolic closure of R_ℓ is contained in the parabolic closure of R. Assume, by contradiction, that the inclusion is proper. Then $\mathit{Span}R_\ell$ is properly included in $\mathit{Span}R$ and hence $R \setminus \mathit{Span}R_\ell \neq \emptyset$. Since R is irreducible, it follows that there exist $\gamma \in R \setminus \mathit{Span}R_\ell$ and $\gamma' \in R_\ell$ such that $\gamma \not\perp \gamma'$. But then $s_\gamma(\gamma')$ is a long root that lies in $R \setminus \mathit{Span}R_\ell$, which is impossible. □

With the same argument, one can also show that, if the roots in R are not all long, then the parabolic closure of R coincides also with the parabolic closure of the set of short roots $R \setminus R_\ell$.

For any $I \subseteq [n]$, we set

$$\Pi_I := \{\alpha_i \in \Pi \mid i \in I\},$$

and

$$\Pi'_I := \{\alpha_i \in \Pi \mid i \notin I\}.$$

Lemma 5.2. *The parabolic subgroup $W(\Pi'_I)$ acts transitively on $(V_I)_\ell$. In particular,*

$$(V_I)_\ell \subseteq \Phi(\{-\theta\} \cup \Pi'_I).$$

Proof. Clearly, $\theta \in (V_I)_\ell$. We prove that any $\gamma \in (V_I)_\ell$ can be transformed into θ by some w in $W(\Pi'_I)$. By contradiction, let γ be a counterexample of maximal height. Then there exists some $\alpha \in \Pi$ such that $(\gamma, \alpha) < 0$ since θ is the unique long root in the closure of the fundamental Weyl chamber, and thus $s_\alpha(\gamma) = \gamma + c\alpha$ with $c > 0$. By definition of V_I, we have that $\alpha \in \Pi'_I$ and $\gamma + c\alpha \in V_I$. By the maximality of $ht(\gamma)$, there exists $w \in W(\Pi'_I)$ such that $\theta = w(\gamma + c\alpha) = ws_\alpha(\gamma)$: a contradiction, since $ws_\alpha \in W(\Pi'_I)$. □

Lemma 5.3. *Let F be a proper parabolic face of $\mathcal{P}_\Phi$. Then F contains at least one long root. Moreover, if F contains some short root, then the ratio between*

the squared lengths of the long and the short roots is 2*, and each short root in* F *is the halfsum of two long roots in* F*.*

Proof. We may assume F standard parabolic, say $F = F_I$, with $I \subseteq [n]$. The first assertion is clear, since θ belongs to any standard parabolic face. If F contains also some short root, then, by Theorem 4.4, there exists two non-orthogonal roots of different lengths, say β and β', in F, with $\frac{(\beta,\beta)}{(\beta',\beta')} = r > 1$. Then $s_{\beta'}\beta = \beta - r\beta'$ is a root and, since $\beta, \beta' \in V_I$, $c_i(s_{\beta'}\beta) = m_i - rm_i$ for all $i \in I$. This implies that $r = 2$, $c_i(s_{\beta'}\beta) = -m_i$, and $-s_{\beta'}\beta = 2\beta' - \beta \in F$. Therefore $\beta' = \frac{1}{2}(\beta - s_{\beta'}\beta)$, a convex combination of the long roots β and $-s_{\beta'}\beta$, both lying in F_I. $\square$

Proposition 5.4. *For any proper standard parabolic face* F*,* $\min V_F$ *is a long root.*

Proof. Set $\eta := \min V_F$ and assume by contradiction that η is short. Then, by Lemma 5.3, η is a convex linear combination of two long roots γ and γ' in V_F. But this implies that $c_i(\eta) \geq \min\{c_i(\gamma), c_i(\gamma')\}$ for all $i \in [n]$, the inequality being strict if $c_i(\gamma) \neq c_i(\gamma')$. Since $\gamma \neq \gamma'$, we obtain a contradiction. $\square$

When considering the extended Dynkin diagram of Φ, we shall call the added node *the affine node* and *finite nodes* the others. Let $I \subseteq [n]$ and consider $\Phi(\{-\theta\} \cup \Pi_I)$. If I is a proper subset of $[n]$, then $\{-\theta\} \cup \Pi_I$ is a root basis for $\Phi(\{-\theta\} \cup \Pi_I)$, and hence the Dynkin diagram of $\Phi(\{-\theta\} \cup \Pi_I)$ is obtained from the extended Dynkin diagram of Φ by removing the nodes corresponding to Π'_I, reading the affine node as $-\theta$.

Let Γ_Φ be the extended Dynkin diagram of Φ, where we read the added node as $-\theta$. For any subset Σ of $\{-\theta\} \cup \Pi$, we denote by $\Gamma(\Sigma)$ the full subdiagram of Γ_Φ having Σ as set of vertices and by $\Gamma_0(\Sigma)$ the connected component of $-\theta$ in $\Gamma(\Sigma)$. Given a subdiagram Γ of a Dynkin diagram, we denote by $\mathrm{v}\Gamma$ the set of vertices of Γ. For any $I \subseteq [n]$, we set

$$\overline{I} := \{k \mid \alpha_k \notin \mathrm{v}\Gamma_0(\{-\theta\} \cup \Pi'_I)\}$$

and

$$\partial I := \{j \mid \alpha_j \in \Pi_I, \text{ and } \exists\, \beta \in \mathrm{v}\Gamma_0(\{-\theta\} \cup \Pi'_I) \text{ s. t. } \beta \not\perp \alpha_j\}.$$

We call $\overline{I}$ the *closure* and ∂I the *border* of I.

Remark 5.1. By definition, $\mathrm{v}\Gamma(\Pi_{\overline{I}})$ is the complement of $\mathrm{v}\Gamma_0(\{-\theta\} \cup \Pi'_I)$ in the vertex set of the extended Dynkin diagram. Hence, $\Gamma_0(\{-\theta\} \cup \Pi'_I) = \Gamma(\{-\theta\} \cup \Pi'_{\overline{I}})$. The border ∂I depends only on $\Gamma_0(\{-\theta\} \cup \Pi'_I)$, hence it is equal to $\partial \overline{I}$ and is the set of the simple roots that are exterior and adjacent to $\Gamma(\{-\theta\} \cup \Pi'_{\overline{I}})$, in the extended Dynkin diagram. In this sense, ∂I is indeed the border of $\overline{I}$.

Lemma 5.5. *Let $I \subseteq [n]$. Then $F_I = F_{\overline{I}}$, $\dim F_I = n - |\overline{I}|$, and $E_{F_I} = Span \Pi'_{\overline{I}}$.*

In particular, for any $J \subseteq [n]$, $F_I = F_J$ if and only if $\Gamma_0(\{-\theta\} \cup \Pi'_I) = \Gamma_0(\{-\theta\} \cup \Pi'_J)$.

Proof. If $I = \emptyset$, the assertion is trivial. Assume $I \neq \emptyset$. By definition, we have that $\Gamma_0(\{-\theta\} \cup \Pi'_I) = \Gamma(\{-\theta\} \cup \Pi'_{\overline{I}})$, in particular, $\Phi(\{-\theta\} \cup \Pi'_{\overline{I}})$ is irreducible: let η be its highest root with respect to the basis $\{\theta\} \cup -\Pi'_{\overline{I}}$. Then, η must be a linear combination of $\{\theta\} \cup -\Pi'_{\overline{I}}$ with positive integral coefficients. It follows that $\eta > 0$ with respect to Π and $c_j(\eta) < m_j$ if and only if $j \in [n] \setminus \overline{I}$. By Theorem 4.1, this implies that $\dim F_{\overline{I}} \geq n - |\overline{I}|$ but, since $\dim F_K \leq n - |K|$ holds for all $K \subseteq [n]$, we have the equality. Clearly, $I \subseteq \overline{I}$, therefore, $F_{\overline{I}} \subseteq F_I$, and $F_{\overline{I}} = F_I$ if and only if $\dim F_{\overline{I}} = \dim F_I$. Assume by contradiction that $\dim F_{\overline{I}} < \dim F_I$ and let $\eta' = \min V_I$, $J = \{j \in [n] \mid c_j(\eta') = m_j\}$. Then, by Theorem 4.1, $I \subseteq J \subsetneq \overline{I}$. Moreover, $\eta' = \min V_J$, therefore, by Proposition 5.4 and Lemma 5.2, $\eta' \in \Phi(\{-\theta\} \cup \Pi'_J)$. By definition, η' has positive coefficients with respect to all roots in the basis $\{\theta\} \cup -\Pi'_J$ of $\Phi(\{-\theta\} \cup \Pi'_J)$: this implies that this must be an irreducible root system. This is impossible, since $\Phi(\{-\theta\} \cup \Pi'_{\overline{I}}) \subsetneq \Phi(\{-\theta\} \cup \Pi'_J) \subseteq \Phi(\{-\theta\} \cup \Pi'_I)$, and by assumption $\Phi(\{-\theta\} \cup \Pi'_{\overline{I}})$ is a maximal irreducible subsystem of $\Phi(\{-\theta\} \cup \Pi'_I)$.

Then, by definition, $E_{F_I} \subseteq Span(\Pi'_{\overline{I}})$. Moreover, both subspaces have dimension $n - |\overline{I}|$. It follows that they are equal.

The second statement follows immediately. □

We explicitly state the following direct consequence.

Theorem 5.6. *The standard parabolic faces of $\mathcal{P}_\Phi$ are in bijection with the connected subdiagrams of the extended Dynkin diagram of Φ that contain the affine node. This bijection is an isomorphism of posets with respect to the inclusions.*

Proof. Immediate from Lemma 5.5. □

Proposition 5.7. *For any $\emptyset \neq I \subseteq [n]$, $\dim F_I = n - |I|$ if and only if $\Phi(\{-\theta\} \cup \Pi'_I)$ is irreducible.*

In this case, $-\min V_I$ is the highest root of $\Phi(\{-\theta\} \cup \Pi'_I)$ with respect to $\{-\theta\} \cup \Pi'_I$, and $\{\min V_I\} \cup \Pi'_I$ is a root basis for $\Phi(\{-\theta\} \cup \Pi'_I)$.

Proof. The first assertion follows directly from Lemma 5.5. Therefore, using Lemma 5.2 and Proposition 5.4, we directly obtain that, if $\dim F_I = n - |I|$ and $\eta = \min V_I$, then $\eta \in \Phi(\{-\theta\} \cup \Pi'_I)$ and $-\eta$ is the highest root of $\Phi(\{-\theta\} \cup \Pi'_I)$ with respect to the basis $\{-\theta\} \cup \Pi'_I$. It is also clear that, with respect to this basis, the coefficient of $-\theta$ in $-\eta$ is 1. By Proposition 2.4, $\{\eta\} \cup \Pi'_I$ is a basis of $\Phi(\{-\theta\} \cup \Pi'_I)$. □

Corollary 5.8. *Let $\emptyset \neq I \subseteq [n]$ and $\eta := \min V_I$. Then V_I coincide with the set of roots of the interval $[\theta, \eta]$ in the root poset of $\Phi(\{-\theta\} \cup \Pi'_I)$ w.r.t. the basis $\{\theta\} \cup -\Pi'_I$. As a subposet of the root poset of Φ, V_I is anti-isomorphic to $[\theta, \eta]$. In particular, $|V_I| = \frac{1}{2}|\Phi(\{-\theta\} \cup \Pi'_I) \setminus \Phi(\Pi'_I)|$.*

Proof. The result follows from Proposition 5.7. □

Proposition 5.9. *For any $I \subseteq [n]$, set*

$$[I] := \{J \subseteq [n] \mid F_J = F_I\}.$$

Then ∂I and $\overline{I}$ are the minimum and the maximum of $[I]$.

Proof. Both assertions follow directly from Lemma 5.5. □

We determine the stabilizer in W of each standard parabolic face, in order to determine its orbit.

Proposition 5.10. *For each $I \subseteq [n]$, let $Stab_W F_I = \{w \in W \mid wF_I = F_I\}$. Then*

$$Stab_W F_I = W(\Pi'_{\partial I}).$$

Proof. It is clear that $Stab_W F_I = Stab_W V_I = \{w \in W \mid wV_I = V_I\}$, therefore it is immediate that $W(\Pi'_I) \subseteq Stab_W F_I$. This should happen for all J such that $F_J = F_I$, in particular for ∂I, therefore $Stab_W F_I \supseteq W(\Pi'_{\partial I})$.

Now, assume by contradiction that $Stab_W F_I \setminus W(\Pi'_{\partial I}) \neq \emptyset$ and let w be an element of minimal length in $Stab_W F_I \setminus W(\Pi'_{\partial I})$. Then $D_r(w) \subseteq \partial I$, therefore there exists $i \in \partial I$ such that $w(\alpha_i) < 0$. Let $\eta = \min V_I$ and $\alpha_i = \alpha$ for short. Then, by Proposition 5.7 and by definition of ∂I, $(\eta, \alpha) > 0$. It follows, by Proposition 2.1, that $\eta - \alpha \in \Phi \cup \{0\}$: on the other hand, $\eta \neq \alpha$ since $\eta \in F_I$ while $\alpha \notin F_I$, because $w(\alpha) \notin F_I$ and w stabilizes F_I. By assumption, $w(\eta) \in V_I$, in particular, $w(\eta) > 0$, therefore $w(\eta - \alpha) = w(\eta) - w(\alpha)$ is a root which is a sum of positive roots, one of which is in F_I. Hence $w(\eta - \alpha) \in F_I$, which is a contradiction since $\eta - \alpha \notin F_I$. □

Remark 5.2. Recall that we denote by w_0 the longest element of W. It is well known that $w_0\Pi = -\Pi$. In particular, for all $\beta \in \Phi$, $ht(\beta) = ht(-w_0\beta)$ and hence that $w_0(\theta) = -\theta$. It follows that, if $w_0(\alpha_i) = -\alpha_{i'}$, then $m_i = m_{i'}$. Moreover, since $Span(\Pi \setminus \{\alpha_i\}) = \breve{\omega}_i^{\perp}$, we have that, if $w_0(\alpha_i) = -\alpha_{i'}$, then $w_0(\breve{\omega}_i)$ is parallel to $\breve{\omega}_{i'}$, and hence $w_0(\breve{\omega}_i) = -\breve{\omega}_{i'}$, since the coweights are in distinct W-orbits.

Theorem 5.11. *All faces of $\mathcal{P}_\Phi$ are parabolic.*

Proof. Let F be a face, and $\dim F = n - p$, with $1 \leq p \leq n$. We prove the claim by induction on p. If $p = 1$, then by Corollary 4.5 and Proposition 2.3,

we get that $\Phi \cap E_F$ is a parabolic root susbsystem of Φ of rank $n-1$. It follows that there exist a $w \in W$ and a $i \in [n]$ such that $wE_F = \breve{\omega}_i^{\perp}$. Therefore, there exists $a \in \mathbb{R}$ such that, for all $\beta \in wF$, $(\beta, \breve{\omega}_i) = a$. This forces $a = \pm m_i$. If $a = m_i$ we obtain that $wF = F_i$. Otherwise, by Remark 5.2, $w_0 wF = F_{i'}$, where $\alpha_{i'} = -w_0\alpha_i$.

Now, we assume $n \geq p > 1$. Let $\tilde{F}$ be any face such that $F \subseteq \tilde{F}$ and $\dim \tilde{F} = n - p + 1$. By induction, we may assume that $\tilde{F}$ is standard parabolic, say $\tilde{F} = F_I$, with $I \subsetneq [n]$, $\overline{I} = I$, and hence that $E_{\tilde{F}} = \mathit{Span}\Pi'_I$ by Lemma 5.5. Then $E_{\tilde{F}} \cap \Phi = \Phi(\Pi'_I)$ (Corollary 4, §1, Ch. VI of [3]), therefore $E_F \cap \Phi = E_F \cap \Phi(\Pi'_I)$. It follows by Proposition 2.3 that $E_F \cap \Phi$ is a parabolic subsystem of $\Phi(\Pi'_I)$ having dimension $n-p$, and hence that there exists a $w \in W(\Pi'_I)$ and a $j \in [n] \setminus I$ such that $wE_F = \mathit{Span}\Pi'_{I\cup\{j\}} = E_{F_I} \cap \breve{\omega}_j^{\perp}$. This implies that $wF = F_I \cap \{x \mid (x, \breve{\omega}_j) = a\}$ for some $a \in \mathbb{R}$ since $wF \subseteq F_I$ ($F \subseteq F_I$ and $wF_I = F_I$ being $w \in W(\Pi'_I) \subseteq \mathit{Stab}_W F_I$ by Proposition 5.10). If we set $\eta = \min V_I$ and $l_j = c_j(\eta)$, arguing as in the case of facets, we obtain that either $a = m_j$ or $a = l_j$. In the first case we are done. Assume that $a = l_j$: this means that $\eta \in wF$ and hence the smallest affine subspace containing wF is $\eta + wE_F = \eta + (E_{F_I} \cap \breve{\omega}_j^{\perp})$. Then, let v be the longest element in $W(\Pi'_I)$. We claim that: (1) $v\eta = \theta$, and (2) there exists a $j' \in [n] \setminus I$ such that $v(E_{F_I} \cap \breve{\omega}_j^{\perp}) = E_{F_I} \cap \breve{\omega}_{j'}^{\perp}$. Let us prove claim (1). By Proposition 5.7, $v(\eta \cup \Pi'_I)$ must be a root basis of $\Phi(\{-\theta\} \cup \Pi'_I)$, and since $v\Pi'_I = -\Pi'_I$, we obtain that $v\eta$ must complete $-\Pi'_I$ to such a root basis. Moreover, $v\eta$ must be in F_I since $W(\Pi'_I)$ stabilizes F_I by Proposition 5.10, and hence in particular is a positive root. It is clear that these two conditions together force $v\eta = \theta$. Then we prove claim (2). For each $k \in [n] \setminus I$, let $\overline{\omega}_k$ be the orthogonal projection of $\breve{\omega}_k$ onto E_{F_I}, so that $E_{F_I} \cap \breve{\omega}_k^{\perp} = E_{F_I} \cap \overline{\omega}_k^{\perp}$. Hence $\{\overline{\omega}_k \mid k \in [n] \setminus I\}$ is the the set of coweights (dual basis) of Π'_I in $\Phi(\Pi'_I)$, and it follows that $v\{\overline{\omega}_k \mid k \in [n] \setminus I\} = -\{\overline{\omega}_k \mid k \in [n] \setminus I\}$ by Remark 5.2. Thus there exists $j' \in [n]$ such that $v(\overline{\omega}_j) = -\overline{\omega}_{j'}$ and since $vE_{F_I} = E_{F_I}$, we obtain that claim (2) holds. Therefore F is parabolic, being $vwF = \theta + \bigcap\limits_{i \in I \cup \{j'\}} \breve{\omega}_i^{\perp}$. $\square$

By Theorems 5.11 and 5.6, we obtain the following classification of the orbits of the faces of $\mathcal{P}_\Phi$. We denote by $\widehat{\Phi}$ the affine root system associated with Φ.

Theorem 5.12. *The orbits of the faces of $\mathcal{P}_\Phi$ under the action of W are in bijection with the connected subdiagrams of the extended Dynkin diagram that contain the affine node. Equivalently, the orbits of the faces are in bijection with the standard parabolic irreducible root subsystems of $\widehat{\Phi}$ that are not included in Φ.*

In particular, the orbits of the facets of $\mathcal{P}_\Phi$ are in bijection with the simple roots that do not disconnect the extended Dynkin diagram, when removed. In Table 1, we list explicitly, type by type, the simple roots corresponding to the standard parabolic facets. In the pictures, the black node corresponds to the affine root, the crossed node and its label denotes the simple root α_i to be removed and its index i.

As a first immediate corollary, we obtain a half-space representation of $\mathcal{P}_\Phi$.

Corollary 5.13. *The polytope $\mathcal{P}_\Phi$ is the intersection of the half-spaces $\{x \mid (x, w\breve{\omega}_i) \leq m_i, \forall w \in W, \forall i \in [n]\}$. A minimal half-space representation is obtained considering only the $i \in [n]$ such that $\widehat{\Pi} \setminus \alpha_i$ is connected.*

Corollary 5.14. *The root polytope $\mathcal{P}_\Phi$ is the convex closure of Φ_ℓ. In particular, $\mathcal{P}_{B_3} = \mathcal{P}_{A_3}$, $\mathcal{P}_{B_n} = \mathcal{P}_{D_n}$ for $n \geq 4$, $\mathcal{P}_{F_4} = \mathcal{P}_{D_4}$, and $\mathcal{P}_{G_2} = \mathcal{P}_{A_2}$.*

Proof. Immediate from Theorem 5.11 and the fact that $F_{[n]} = \{\theta\}$. □

We explicitly notice that $\mathcal{P}_{C_n}$ is the cross-polytope for all $n \geq 2$ (octahedron for $n = 3$), and that $\mathcal{P}_{A_n}$ and $\mathcal{P}_{B_n} = \mathcal{P}_{D_n}$, for $n \geq 4$, are distinct n-dimensional generalizations of the cuboctahedron $\mathcal{P}_{A_3} = \mathcal{P}_{B_3}$ (see [6]).

Corollary 5.15. *Let $\mathcal{I} := \{\Gamma \subseteq \Pi \mid \Phi(\widehat{\Gamma})$ is irreducible$\}$. The f-polynomial of $\mathcal{P}_\Phi$ is*

$$\sum_{\Gamma \in \mathcal{I}} [W : W_{\Gamma^*}] t^{|\Gamma|}.$$

Proof. It follows by Lemma 5.5 , Proposition 5.10, and Theorem 5.11. □

If Φ is not simply laced, then the short roots are never vertices of $\mathcal{P}_\Phi$. The following result tells how far they are from being vertices.

Proposition 5.16. *Let Φ be not simply laced, θ_s be the highest short root of Φ, and $I := \{i \in [n] \mid c_i(\theta_s) = m_i\}$. Then the minimal dimension of the faces of $\mathcal{P}_\Phi$ containing a short root is $n - |I|$. Moreover, the faces of dimension $n - |I|$ containing a short root form a single W-orbit.*

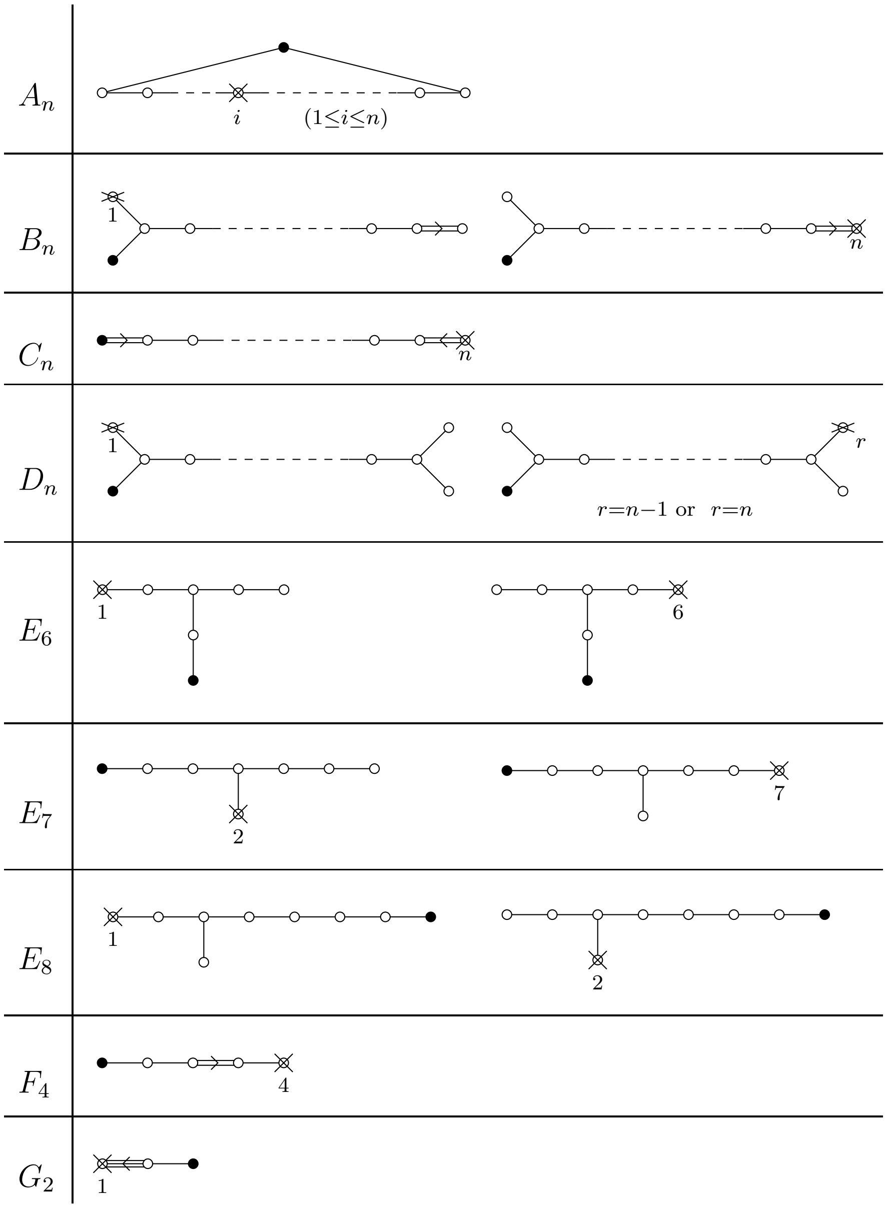
Proof. Any standard parabolic face F containing a short root contains θ_s, since V_F is a dual order ideal in the root poset and θ_s is greater than any other short root. By Theorem 5.11, it is enough to find the dimension of the minimal standard parabolic face of $\mathcal{P}_\Phi$ containing θ_s. This is clearly F_I, and its dimension is $n - |I|$ by Theorem 4.1. The last statement follows since the action of W is transitive on the set of short roots and the intersection of two distinct faces of dimension $n - |I|$ does not contain any short root. □

Suppose that Φ be not simply laced and let $I := \{i \in [n] \mid c_i(\theta_s) = m_i\}$. By Lemma 5.3, we already know that, if the ratio between the squared lengths of the long and the short roots is 3, then the short roots lie in the inner part of $\mathcal{P}_\Phi$, and hence I must be empty. On the other hand, if the ratio between the squared lengths of the long and the short roots is 2, by a case-by-case argument we see that $I \neq \emptyset$, and hence the short roots are always on the border of $\mathcal{P}_\Phi$.

6. The ideals of the faces as elements of the affine Weyl group

By Theorem 5.11, every face F of $\mathcal{P}_\Phi$ is a standard parabolic face, for an appropriate choice of the basis Π. Therefore, by Proposition 3.1 and Theorem

Table 1. Subdiagrams corresponding to the standard parabolic facets



4.1, each face is associated with a principal abelian ideal of a Borel subalgebra. Theorem 5.12 gives a natural description of $\mathcal{P}_\Phi$ in terms of the affine root system associated with Φ. Also the ideals associated with the faces of $\mathcal{P}_\Phi$ have a natural description as special elements of the affine Weyl group. In this section, we recall briefly some basic facts about the affine root system associated with Φ and about Peterson's map from the abelian ideals of the Borel subalgebra to the affine Weyl group, in order to make explicit these elements.

Let $\widehat{\Phi}$ be the affine root system associated with Φ and $\widehat{W}$ be its Weyl group. For the general theory of affine root systems we refer to [11]. Here we give a minimal description, which is enough for our purposes. Indeed, according to [11], our $\widehat{\Phi}$ is the set of real roots of the untwisted affine root system associated with Φ. (See also [5] or [4] for few more details). As usual, we denote by δ the basic imaginary root, so that $Span_{\mathbb{R}}(\widehat{\Phi}) = Span_{\mathbb{R}}\Phi \oplus \mathbb{R}\delta$ and

$$\widehat{\Phi} = \Phi + \mathbb{Z}\delta := \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}.$$

This is an affine root system in $Span_{\mathbb{R}}(\widehat{\Phi})$ endowed with the bilinear form obtained by extending the scalar product of $Span_{\mathbb{R}}\Phi$ to a positive semidefinite form with kernel $\mathbb{R}\delta$. If we take $\alpha_0 = -\theta + \delta$, then $\widehat{\Pi} := \{\alpha_0\} \cup \Pi$ is a root basis for $\widehat{\Phi}$. The set of positive roots of $\widehat{\Phi}$ with respect to $\widehat{\Pi}$ is $\widehat{\Phi}^+ := \Phi^+ \cup (\Phi + \mathbb{Z}^+\delta)$, where $\mathbb{Z}^+$ is the set of positive integers.

Let $\mathfrak{i}$ be an abelian ideal of $\mathfrak{b}$, $V_\mathfrak{i} := \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}$, and $-V_\mathfrak{i} + \delta := \{-\alpha + \delta \mid \alpha \in V_\mathfrak{i}\}$. For all $w \in \widehat{W}$, set $N(w) := \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) < 0\}$. It is is well known that $N(w)$ uniquely determines w. Moreover, for all $v, w \in \widehat{W}$, we have that $N(vw) = (N^\pm(v) \triangle vN^\pm(w)) \cap \widehat{\Phi}^+$, where, for all $S \subseteq \widehat{\Phi}^+$, $S^\pm$ stands for $S \cup -S$ and $\triangle$ denotes the symmetric difference. Then, there exists a unique element $w_\mathfrak{i} \in \widehat{W}$ such that $N(w_\mathfrak{i}) = -V_\mathfrak{i} + \delta$. Moreover, the map $\mathfrak{i} \mapsto w_\mathfrak{i}$ is a bijection from the set of the abelian ideals of $\mathfrak{b}$ and the set of all elements w in $\widehat{W}$ such that $N(w) \subseteq -\Phi^+ + \delta$ (Peterson, [12]).

We extend the notation $\Phi(\Sigma)$ and $\Phi^+(\Sigma)$ to the subsets Σ of $\widehat{\Pi}$, so $\widehat{\Phi}(\Sigma)$ denotes the subsystem of $\widehat{\Phi}$ generated by Σ and $\widehat{\Phi}^+(\Sigma) := \widehat{\Phi}(\Sigma) \cap \widehat{\Phi}^+$. When $\Sigma \subseteq \Pi$, $\widehat{\Phi}(\Sigma) = \Phi(\Sigma)$ and $\widehat{\Phi}^+(\Sigma) = \Phi^+(\Sigma)$. For any proper subset Γ of Π, the root system $\widehat{\Phi}(\{\alpha_0\} \cup \Gamma)$ is naturally isomorphic, through the projection onto $Span_{\mathbb{R}}\Phi$, to the root system $\Phi(\{-\theta\} \cup \Gamma)$. And, through this bijection, $\widehat{\Phi}^+(\{\alpha_0\} \cup \Gamma) \setminus \widehat{\Phi}^+(\Gamma)$ corresponds to $-(\Phi^+(\{-\theta\} \cup \Gamma) \setminus \Phi^+(\Gamma))$. Therefore, by Corollary 5.8, for any nonempty $I \subseteq \Pi$, we obtain that

$$-V_I + \delta = \widehat{\Phi}^+(\{\alpha_0\} \cup \Pi'_{\overline{I}}) \setminus \widehat{\Phi}^+(\Pi'_{\overline{I}}).$$

Thus the element of the affine Weyl group that corresponds to the abelian ideal $\mathfrak{i}_{V_I}$ can be easily determined. In fact, if we denote by $\widehat{w}'_{0\overline{I}}$ the longest element in

$\widehat{W}(\{\alpha_0\} \cup \Pi'_{\overline{I}})$ and by $w'_{0\overline{I}}$ the longest element in $W(\Pi'_{\overline{I}})$, we obtain that

$$w_{\mathfrak{i}_{V_I}} = \widehat{w}'_{0\overline{I}} w'_{0\overline{I}}.$$

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